

SECTION—A

1. (a) Let $V = \mathbb{R}^4$. Find a basis and dimension of the subspace

$$W = \{(a, b, c, d) \in V : a = b + c, c = b + d\}$$

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- (b) Describe explicitly a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, which has its range spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

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- (c) Find the relation between the radii of a right circular cylinder and a cone if the former with maximum possible curved surface area is inscribed in the latter.

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- (d) Find the limit of $(\cot x - \tan x)^{\frac{1}{\log_e x}}$, when $x \rightarrow 0$.

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- (e) Show that if $ax^2 + 2hxy + by^2 + 2gx + 1 = 0$ represents two straight lines, then $b < 0$ and $bg^2 + h^2 = ab$.

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2. (a) Let $W_1 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$ and $W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{C} \right\}$ be two subspaces of the vector space of all 2×2 matrices over the complex field $\mathbb{C}$. Show that

$$\dim \left(\frac{W_1 + W_2}{W_2} \right) = \dim \left(\frac{W_1}{W_1 \cap W_2} \right)$$

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- (b) Evaluate the volume of the solid formed by rotating the curve $r = a(1 + \cos \theta)$ about the initial line.

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- (c) (i) Reduce the equation $(c^2 + d^2)(x^2 + y^2) = (cx + dy + 2f)^2$ to its canonical form and show that it represents a parabola. Find the latus rectum of the parabola.

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- (ii) A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the lines

$$y - x \tan \theta = 0 = z - c$$

$$y + x \tan \theta = 0 = z + c$$

in the points P and Q . If $|PQ| = 2a$ (where a is a +ve number), then show that the centre of all such spheres lies on the circle $x^2 + y^2 = (a^2 - c^2) \operatorname{cosec}^2 2\theta$, $z = 0$.

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3. (a) If $u = \exp\left\{\sin^{-1} \frac{x+y}{\sqrt{x}-\sqrt{y}}\right\}$, then show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u \tan(\log_e u)$. 10

(b) Reduce the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}$$

to echelon form and then to row canonical form. 15

- (c) (i) Show that the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1, y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$. Further show that if $2d$ is the shortest distance between the given lines, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2} \quad 8$$

- (ii) A variable plane is parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the coordinate axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0 \quad 7$$

4. (a) Find the equations of the generating lines of the hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

passing through the point $(2, 3, -4)$. 10

- (b) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$T(x, y, z) = (5x - y + 3z, -6x + 4y - 6z, -6x + 2y - 4z)$$

Find all the eigenvalues and corresponding eigenvectors. 15

- (c) (i) How many loops are generated of the curve $r = a \sin 3\theta$? Find the sum of the areas of all the loops. 8

- (ii) Deduce the asymptote of the curve $r \log_e \theta = a$. 7

SECTION—B

5. (a) Solve the differential equation

$$p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0, \text{ where } p = \frac{dy}{dx} \quad 8$$

- (b) Solve the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cosh x$. 8

- (c) A particle moves from rest at a distance a from a centre of force where repulsion at distance x is μx^{-2} . Show that its velocity at distance x is

$$\sqrt{\frac{2\mu(x-a)}{ax}}$$

and that the time it has taken is

$$\sqrt{\frac{a}{2\mu}} \left[\sqrt{x^2 - ax} + a \log_e \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a} - 1} \right) \right] \quad 8$$

- (d) Two uniform steel rods of equal size l hang from their junction and rest on a symmetrically placed smooth vertical circular base of radius a . If each of the rods subtends an angle ϕ with the vertical line passing through the centre of the circular base, show that, applying the principle of virtual work, the relation obtained is $l = 2a \cot \phi \operatorname{cosec}^2 \phi$. 8

- (e) If $\vec{F}$ is a solenoidal vector, then show that

$$\operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} \vec{F} = \nabla^2 \nabla^2 \vec{F} = \nabla^4 \vec{F} \quad 8$$

6. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 29y = xe^{5x} + \sin 2x \quad 10$$

- (b) A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to $\frac{\mu}{(\text{distance})^2}$. Show that its path is a conic section and distinguish between the three cases that arise. Further show that the square of the periodic time varies as the cube of the major axis. 15

- (c) (i) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\operatorname{curl} \left(\frac{\vec{r}}{|\vec{r}|} \right)$. 6

- (ii) Find the curvature and torsion of the curve $x = a \cos t$, $y = a \sin t$, $z = bt$. 9

7. (a) If near the surface of a celestial body having atmosphere, the gravity is almost constant and the absolute temperature in its atmosphere is given by

$$T = T_0 \sqrt{1 - \frac{z^2}{n^2 H^2}}$$

H being the height of the homogeneous atmosphere and n a constant quantity, show that the pressure in the atmosphere will be given by

$$p = p_0 \exp\left(\sin^{-1} \frac{z}{nH}\right)$$

where p_0 is the pressure at $z = 0$.

- (b) Solve the differential equation

$$\frac{d^2 y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$$

- (c) (i) If ϕ satisfies $\nabla^2 \phi = 0$, then show that $\nabla \phi$ is both solenoidal and irrotational.
(ii) Verify the divergence theorem for the function

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$$

taken over the parallelepiped $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

8. (a) Verify Green's theorem for

$$\int_C [(xy + y^2)dx + x^2 dy]$$

where C is bounded by the curves $y = x$ and $y = x^2$.

- (b) Using the method of variation of parameters, solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 \log x, \quad x > 0$$

- (c) Establish a stability criterion if a rigid body is lying on another rigid body at a point of contact, and also both have rough surfaces to prevent sliding and a small area around the point of contact of both of them is circular.

A solid frustum of a paraboloid of revolution of height h and latus rectum $2a$ rests with its vertex on that of a paraboloid of revolution of latus rectum $2b$. Find the stability condition.